

INVARIANTS OF COHEN-MACAULAY RINGS ASSOCIATED TO THEIR CANONICAL IDEALS

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ABSTRACT. The purpose of this paper is to introduce new invariants of Cohen-Macaulay local rings. Our focus is the class of Cohen-Macaulay local rings that admit a canonical ideal. Attached to each such ring \mathbf{R} with a canonical ideal \mathcal{C} , there are integers—the type of \mathbf{R} , the reduction number of \mathcal{C} —that provide valuable metrics to express the deviation of \mathbf{R} from being a Gorenstein ring. We enlarge this list with other integers—the roots of \mathbf{R} and several canonical degrees. The latter are multiplicity based functions of the Rees algebra of \mathcal{C} .

Key Words and Phrases: Canonical degree, Cohen-Macaulay type, analytic spread, roots, reduction number.

1. INTRODUCTION

Let \mathbf{R} be a Cohen-Macaulay ring of dimension $d \geq 1$. If \mathbf{R} admits a canonical module \mathcal{C} and has a Gorenstein total ring of fractions, we may assume that \mathcal{C} is an ideal of \mathbf{R} . In this case, we introduce new numerical invariants for \mathbf{R} that refine and extend for local rings the use of its Cohen-Macaulay *type*, the minimal number of generators of \mathcal{C} , $r(\mathbf{R}) = \nu(\mathcal{C})$. Among numerical invariants of the isomorphism class of \mathcal{C} are the *analytic spread* $\ell(\mathcal{C})$ of \mathcal{C} and attached reduction numbers. We also introduce the *rootset* of \mathbf{R} , which may be a novel invariant. Certain constructions on \mathcal{C} , such as the Rees algebra $\mathbf{R}[\mathcal{C}T]$ leads to an invariant of \mathbf{R} , but the associated graded ring $\text{gr}_{\mathcal{C}}(\mathbf{R})$ does not. It carries however properties of a semi-invariant which we will make use of to build true invariants. Combinations of semi-invariants are then used to build invariants under the general designation of *canonical degrees*. The main effort is setting the foundations of the new invariants and examining their relationships. We shall also experiment in extending the construction to more general rings. When we do so, to facilitate the discussion we assume that \mathbf{R} is a homomorphic image of a Gorenstein ring. In a sequel we make applications to Rees algebras, monomial rings, Stanley-Reisner rings.

In Section 2 we introduce our basic canonical degree and derive some of its most direct properties. It requires knowledge of the Hilbert coefficients $e_0(\cdot)$ of \mathfrak{m} -primary ideals:

Theorem 2.2. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ that has a canonical ideal \mathcal{C} . Then

$$\text{cdeg}(\mathbf{R}) = \sum_{\text{height } \mathfrak{p}=1} \text{cdeg}(\mathbf{R}_{\mathfrak{p}}) \deg(\mathbf{R}/\mathfrak{p}) = \sum_{\text{height } \mathfrak{p}=1} [e_0(\mathcal{C}_{\mathfrak{p}}) - \lambda((\mathbf{R}/\mathcal{C})_{\mathfrak{p}})] \deg(\mathbf{R}/\mathfrak{p})$$

is a well-defined finite sum independent of the chosen canonical ideal \mathcal{C} . In particular, if \mathcal{C} is equimultiple with a minimal reduction (a) , then

$$\text{cdeg}(\mathbf{R}) = \deg(\mathcal{C}/(a)) = e_0(\mathfrak{m}, \mathcal{C}/(a)).$$

Two of its consequences when \mathcal{C} is equimultiple are: (i) $\text{cdeg}(\mathbf{R}) \geq r(\mathbf{R}) - 1$; (ii) $\text{cdeg}(\mathbf{R}) = 0$ if and only if \mathbf{R} is Gorenstein.

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In Section 3, the minimal value for $\text{cdeg}(\cdot)$ is assumed on a new class of Cohen-Macaulay rings, called *almost Gorenstein rings*, introduced in [2], and developed in [11] and [13]:

Proposition 3.2. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ that has a canonical ideal \mathcal{C} . If \mathcal{C} is equimultiple and $\text{cdeg}(\mathbf{R}) = r(\mathbf{R}) - 1$, then there is an exact sequence of \mathbf{R} -modules $0 \rightarrow \mathbf{R} \rightarrow \mathcal{C} \rightarrow X \rightarrow 0$ such that $\nu(X) = e_0(X)$.

The next three sections carry out several general calculations seeking relations between $\text{cdeg}(\mathbf{R})$ and other invariants of \mathbf{R} in the case of low Cohen-Macaulay type. For instance, it treats the notion of the *rootset* of \mathbf{R} , made up of the positive integers n such that $L^n \simeq \mathcal{C}$ for some fractional ideal L . In dimension 1 this is a finite set of cardinality $\leq r(\mathbf{R}) - 1$. We also describe the rootset for fairly general monomial rings. Section 6 deals with general properties of $\text{cdeg}(\cdot)$ under a change of rings: polynomial rings, completion and hyperplane section. It has an extended but quick treatment of the changes of $\text{cdeg}(\mathbf{A})$ when \mathbf{A} is the ring obtained by augmenting \mathbf{R} by a module.

In Section 7 we discuss extensions of the canonical degree. A clear shortcoming of the definition of $\text{cdeg}(\mathbf{R})$, e.g. that be independent of the canonical ideal and that its vanishing is equivalent to the Gorenstein property, is that it does not seem to work for rings of dimension ≥ 2 . A natural choice would be for $G = \text{gr}_{\mathcal{C}}(\mathbf{R})$ to define the *canonical degree** of \mathbf{R} as the integer $\text{cdeg}_{\mathcal{C}}^*(\mathbf{R}) = \deg(\text{gr}_{\mathcal{C}}(\mathbf{R})) - \deg(\mathbf{R}/\mathcal{C})$. Finally, an appropriate hyperplane section readily creates a degree that has the Gorenstein property but we were unable to prove the independence property.

2. CANONICAL DEGREE

For basic references on canonical modules we will use [5], [4] and [14], while for the existence and properties of canonical ideals we use [1]. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring. Suppose that \mathbf{R} has a canonical ideal \mathcal{C} . In this setting we introduce a numerical degree for \mathbf{R} and study its properties. The starting point of our discussion is the following elementary observation. We denote the length by λ .

Proposition 2.1. Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Then the integer $\text{cdeg}(\mathbf{R}) = e_0(\mathcal{C}) - \lambda(\mathbf{R}/\mathcal{C})$ is independent of the canonical ideal \mathcal{C} .

Proof. If x is an indeterminate over \mathbf{R} , in calculating these differences we may pass from \mathbf{R} to $\mathbf{R}(x) = \mathbf{R}[x]_{\mathfrak{m}\mathbf{R}[x]}$, in particular we may assume that the ring has an infinite residue field.

Let \mathcal{C} and \mathcal{D} be two canonical ideals. Suppose (a) is a minimal reduction of \mathcal{C} . Since $\mathcal{D} \simeq \mathcal{C}$ ([5, Theorem 3.3.4]), $\mathcal{D} = q\mathcal{C}$ for some fraction q . If $\mathcal{C}^{n+1} = (a)\mathcal{C}^n$ by multiplying it by q^{n+1} , we get $\mathcal{D}^{n+1} = (qa)\mathcal{D}^n$, where $(qa) \subset \mathcal{D}$. Thus (qa) is a reduction of \mathcal{D} and $\mathcal{C}/(a) \simeq \mathcal{D}/(qa)$. Taking their co-lengths we have

$$\lambda(\mathbf{R}/(a)) - \lambda(\mathbf{R}/\mathcal{C}) = \lambda(\mathbf{R}/(qa)) - \lambda(\mathbf{R}/\mathcal{D}).$$

Since $\lambda(\mathbf{R}/(a)) = e_0(\mathcal{C})$ and $\lambda(\mathbf{R}/(qa)) = e_0(\mathcal{D})$, we have

$$e_0(\mathcal{C}) - \lambda(\mathbf{R}/\mathcal{C}) = e_0(\mathcal{D}) - \lambda(\mathbf{R}/\mathcal{D}).$$

□

We can define $\text{cdeg}(\mathbf{R})$ in full generality as follows.

Theorem 2.2. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ that has a canonical ideal \mathcal{C} . Then

$$\text{cdeg}(\mathbf{R}) = \sum_{\text{height } \mathfrak{p}=1} \text{cdeg}(\mathbf{R}_{\mathfrak{p}}) \deg(\mathbf{R}/\mathfrak{p}) = \sum_{\text{height } \mathfrak{p}=1} [e_0(\mathcal{C}_{\mathfrak{p}}) - \lambda((\mathbf{R}/\mathcal{C})_{\mathfrak{p}})] \deg(\mathbf{R}/\mathfrak{p})$$

is a well-defined finite sum independent of the chosen canonical ideal \mathcal{C} . In particular, if \mathcal{C} is equimultiple with a minimal reduction (a) , then

$$\text{cdeg}(\mathbf{R}) = \deg(\mathcal{C}/(a)) = e_0(\mathbf{m}, \mathcal{C}/(a)).$$

Proof. By Proposition 2.1, the integer $\text{cdeg}(\mathbf{R}_{\mathfrak{p}})$ does not depend on the choice of a canonical ideal of \mathbf{R} . Also $\text{cdeg}(\mathbf{R})$ is a finite sum since, if $\mathfrak{p} \notin \text{Min}(\mathcal{C})$, then $\mathcal{C}_{\mathfrak{p}} = \mathbf{R}_{\mathfrak{p}}$ so that $\mathbf{R}_{\mathfrak{p}}$ is Gorenstein. Thus $\text{cdeg}(\mathbf{R}_{\mathfrak{p}}) = 0$. The last assertion follows from the associativity formula:

$$\text{cdeg}(\mathbf{R}) = \sum_{\text{height } \mathfrak{p}=1} \lambda((\mathcal{C}/(a))_{\mathfrak{p}}) \deg(\mathbf{R}/\mathfrak{p}) = \deg(\mathcal{C}/(a)).$$

□

Definition 2.3. Let (\mathbf{R}, \mathbf{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$ that has a canonical ideal. Then the *canonical degree* of \mathbf{R} is the integer

$$\text{cdeg}(\mathbf{R}) = \sum_{\text{height } \mathfrak{p}=1} \text{cdeg}(\mathbf{R}_{\mathfrak{p}}) \deg(\mathbf{R}/\mathfrak{p}).$$

Corollary 2.4. $\text{cdeg}(\mathbf{R}) \geq 0$ and vanishes if and only if \mathbf{R} is Gorenstein in codimension 1.

Corollary 2.5. Suppose that the canonical ideal of \mathbf{R} is equimultiple. Then we have the following.

- (1) $\text{cdeg}(\mathbf{R}) \geq r(\mathbf{R}) - 1$.
- (2) $\text{cdeg}(\mathbf{R}) = 0$ if and only if \mathbf{R} is Gorenstein.

Proof. Let (a) be a minimal reduction of the canonical ideal \mathcal{C} . Then

$$\text{cdeg}(\mathbf{R}) = e_0(\mathbf{m}, \mathcal{C}/(a)) \geq \nu(\mathcal{C}/(a)) = r(\mathbf{R}) - 1.$$

If $\text{cdeg}(\mathbf{R}) = 0$ then $r(\mathbf{R}) = 1$, which proves that \mathbf{R} is Gorenstein. □

Now we extend the above result to a more general class of ideals when the ring has dimension one. We recall that if \mathbf{R} is a 1-dimensional Cohen-Macaulay local ring, I is an \mathbf{m} -primary ideal with minimal reduction (a) , then the reduction number of I relative to (a) is independent of the reduction ([15, Theorem 1.2]). It will be denoted simply by $\text{red}(I)$.

Proposition 2.6. Let (\mathbf{R}, \mathbf{m}) be a 1-dimensional Cohen-Macaulay local ring which is not a valuation ring. Let I be an irreducible \mathbf{m} -primary ideal such that $I \subset \mathbf{m}^2$. If (a) is a minimal reduction of I , then $\lambda(I/(a)) \geq r(\mathbf{R}) - 1$. In the case of equality, $\text{red}(I) \leq 2$.

Proof. Let $L = I : \mathbf{m}$ and $N = (a) : \mathbf{m}$. Then $\lambda(L/I) = r(\mathbf{R}/I) = 1$ and $\lambda(N/(a)) = r(\mathbf{R})$. Thus, we have

$$r(\mathbf{R}) \leq \lambda(L/N) + \lambda(N/(a)) = \lambda(L/(a)) = \lambda(L/I) + \lambda(I/(a)) = 1 + \lambda(I/(a)),$$

which proves that $\lambda(I/(a)) \geq r(\mathbf{R}) - 1$.

Suppose that $\lambda(I/(a)) = r(\mathbf{R}) - 1$. Then $L = N$. By [6, Lemma 3.6], L is integral over I . Thus, L is integral over (a) . By [7, Theorem 2.3], $\text{red}(N) = 1$. Hence $L^2 = aL$. Since $\lambda(L/I) = 1$, by [12, Proposition 2.6], we have $I^3 = aI^2$. □

3. EXTREMAL VALUES OF THE CANONICAL DEGREE

We examine in this section extremal values of the canonical degree. First we recall the definition of almost Gorenstein rings ([2, 11, 13]).

Definition 3.1. ([13, Definition 3.3]) A Cohen-Macaulay local ring \mathbf{R} with a canonical module $K_{\mathbf{R}}$ is said to be an *almost Gorenstein* ring if there exists an exact sequence of \mathbf{R} -modules $0 \rightarrow \mathbf{R} \rightarrow K_{\mathbf{R}} \rightarrow X \rightarrow 0$ such that $\nu(X) = e_0(X)$.

Proposition 3.2. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Assume that \mathcal{C} is equimultiple. If $\text{cdeg}(\mathbf{R}) = r(\mathbf{R}) - 1$, then \mathbf{R} is an almost Gorenstein ring. In particular, if $\text{cdeg}(\mathbf{R}) \leq 1$, then \mathbf{R} is an almost Gorenstein ring.*

Proof. We may assume that \mathbf{R} is not a Gorenstein ring. Let (a) be a minimal reduction of \mathcal{C} . Consider the exact sequence of \mathbf{R} -modules

$$0 \rightarrow \mathbf{R} \xrightarrow{\varphi} \mathcal{C} \rightarrow X \rightarrow 0, \text{ where } \varphi(1) = a.$$

Then $\nu(X) = r(\mathbf{R}) - 1 = \text{cdeg}(\mathbf{R}) = e_0(X)$. Thus, \mathbf{R} is an almost Gorenstein ring. \square

Proposition 3.3. *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Then $\text{cdeg}(\mathbf{R}) = r(\mathbf{R}) - 1$ if and only if \mathbf{R} is an almost Gorenstein ring.*

Proof. It is enough to prove that the converse holds true. We may assume that \mathbf{R} is not a Gorenstein ring. Let (a) be a minimal reduction of \mathcal{C} . Since \mathbf{R} is almost Gorenstein, there exists an exact sequence of \mathbf{R} -modules

$$0 \rightarrow \mathbf{R} \xrightarrow{\psi} \mathcal{C} \rightarrow Y \rightarrow 0 \text{ such that } \nu(Y) = e_0(Y).$$

Since $\dim(Y) = 0$ by [13, Lemma 3.1], we have that $\mathfrak{m}Y = (0)$. Let $b = \psi(1) \in \mathcal{C}$ and set $\mathfrak{q} = (b)$. Then $\mathfrak{m}\mathfrak{q} \subseteq \mathfrak{m}\mathcal{C} \subseteq \mathfrak{q}$. Therefore, since \mathbf{R} is not a DVR and $\lambda(\mathfrak{q}/\mathfrak{m}\mathfrak{q}) = 1$, we get $\mathfrak{m}\mathcal{C} = \mathfrak{m}\mathfrak{q}$, whence the ideal \mathfrak{q} is a reduction of \mathcal{C} by the Cayley-Hamilton theorem, so that $\text{cdeg}(\mathbf{R}) = e_0(Y) = \nu(Y) = r(\mathbf{R}) - 1$. \square

Now we consider the general case when a canonical ideal is not necessarily equimultiple.

Lemma 3.4. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field and a canonical ideal \mathcal{C} . Let a be an element of \mathcal{C} such that (i) for $\forall \mathfrak{p} \in \text{Ass}(\mathbf{R}/\mathcal{C})$ the element $\frac{a}{1}$ generates a reduction of $\mathcal{C}\mathbf{R}_{\mathfrak{p}}$, (ii) a is \mathbf{R} -regular, and (iii) $a \notin \mathfrak{m}\mathcal{C}$. Let $Z = \{\mathfrak{p} \in \text{Ass}(\mathcal{C}/(a)) \mid \mathcal{C} \not\subseteq \mathfrak{p}\}$.*

$$(1) \text{ cdeg}(\mathbf{R}) = \deg(\mathcal{C}/(a)) - \sum_{\mathfrak{p} \in Z} \lambda((\mathcal{C}/(a))_{\mathfrak{p}}) \deg(\mathbf{R}/\mathfrak{p}).$$

$$(2) \text{ cdeg}(\mathbf{R}) = \deg(\mathcal{C}/(a)) \text{ if and only if } \text{Ass}(\mathcal{C}/(a)) \subseteq V(\mathcal{C}).$$

Proof. It follows from Theorem 2.2 and $\deg(\mathcal{C}/(a)) = \sum_{\text{height } \mathfrak{p}=1} \lambda((\mathcal{C}/(a))_{\mathfrak{p}}) \deg(\mathbf{R}/\mathfrak{p})$. \square

Theorem 3.5. *With the same notation given in Lemma 3.4, suppose that $\text{Ass}(\mathcal{C}/(a)) \subseteq V(\mathcal{C})$. Then $\text{cdeg}(\mathbf{R}) = r(\mathbf{R}) - 1$ if and only if \mathbf{R} is an almost Gorenstein ring.*

Proof. It is enough to prove the converse holds true. We may assume that \mathbf{R} is not a Gorenstein ring. Choose an exact sequence

$$0 \rightarrow \mathbf{R} \rightarrow \mathcal{C} \rightarrow Y \rightarrow 0$$

such that $\deg(Y) = \nu(Y) = r(\mathbf{R}) - 1$. Since $\text{Ass}(\mathcal{C}/(a)) \subseteq V(\mathcal{C})$, we have $\deg(Y) \geq \deg(\mathcal{C}/(a))$. By Lemma 3.4 and the proof of Corollary 2.5, we obtain the following:

$$r(\mathbf{R}) - 1 \leq \text{cdeg}(\mathbf{R}) = \deg(\mathcal{C}/(a)) \leq \deg(Y) = r(\mathbf{R}) - 1.$$

\square

Remark 3.6. If \mathbf{R} is a non-Gorenstein normal domain, then its canonical ideal cannot be equimultiple.

Proof. Suppose that a canonical ideal \mathcal{C} of \mathbf{R} is equimultiple, i.e., $\mathcal{C}^{n+1} = a\mathcal{C}^n$. Then we would have an equation $(n+1)[\mathcal{C}] = n[\mathcal{C}]$ in its divisor class group. This means that $[\mathcal{C}] = [0]$. Thus, $\mathcal{C} \simeq \mathbf{R}$. Hence \mathcal{C} can not be equimultiple. \square

4. CANONICAL INDEX

Throughout the section, let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field and suppose that a canonical ideal \mathcal{C} exists. We begin by showing that the reduction number of a canonical ideal of \mathbf{R} is an invariant of the ring.

Proposition 4.1. *Let \mathcal{C} and \mathcal{D} be canonical ideals of \mathbf{R} . Then $\text{red}(\mathcal{C}) = \text{red}(\mathcal{D})$.*

Proof. Let K be the total ring of quotients of \mathbf{R} . Then there exists $q \in K$ such that $\mathcal{D} = q\mathcal{C}$. Let $r = \text{red}(\mathcal{C})$ and J a minimal reduction of \mathcal{C} with $\mathcal{C}^{r+1} = J\mathcal{C}^r$. Then

$$\mathcal{D}^{r+1} = (q\mathcal{C})^{r+1} = q^{r+1}(J\mathcal{C}^r) = (qJ)(q\mathcal{C})^r = qJ\mathcal{D}^r$$

so that $\text{red}(\mathcal{D}) \leq \text{red}(\mathcal{C})$. Similarly, $\text{red}(\mathcal{C}) \leq \text{red}(\mathcal{D})$. \square

Definition 4.2. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with a canonical ideal \mathcal{C} . The *canonical index* of \mathbf{R} is the reduction number of the canonical ideal \mathcal{C} of \mathbf{R} and is denoted by $\rho(\mathbf{R})$.

Remark 4.3. Suppose that \mathbf{R} is not Gorenstein. The following are known facts.

- (1) If the canonical ideal of \mathbf{R} is equimultiple, then $\rho(\mathbf{R}) \neq 1$.
- (2) If $\dim \mathbf{R} = 1$ and $e_0(\mathfrak{m}) = 3$, then $\rho(\mathbf{R}) = 2$.
- (3) If $\dim \mathbf{R} = 1$ and $\text{cdeg}(\mathbf{R}) = r(\mathbf{R}) - 1$, then $\rho(\mathbf{R}) = 2$.

Proof. (1) Suppose that $\rho(\mathbf{R}) = 1$. Let \mathcal{C} be a canonical ideal of \mathbf{R} with $\mathcal{C}^2 = a\mathcal{C}$. Then $\mathcal{C}a^{-1} \subset \text{Hom}(\mathcal{C}, \mathcal{C}) = \mathbf{R}$ so that $\mathcal{C} = (a)$. This is a contradiction.

(2) It follows from the fact that, if $(\mathbf{R}, \mathfrak{m})$ is a 1-dimensional Cohen-Macaulay ring and I an \mathfrak{m} -primary ideal, then $\text{red}(I) \leq e_0(\mathfrak{m}) - 1$.

(3) It follows from Proposition 3.3 and [11, Theorem 3.16]. \square

Sally module. We examine briefly the Sally module associated to the canonical ideal \mathcal{C} in rings of dimension 1. Let $Q = (a)$ be a minimal reduction of \mathcal{C} and consider the exact sequence of finitely generated $\mathbf{R}[Q\mathbf{T}]$ -modules

$$0 \rightarrow \mathcal{C}\mathbf{R}[Q\mathbf{T}] \rightarrow \mathcal{C}\mathbf{R}[\mathcal{C}\mathbf{T}] \rightarrow S_Q(\mathcal{C}) \rightarrow 0.$$

Then the Sally module $S = S_Q(\mathcal{C}) = \bigoplus_{n \geq 1} \mathcal{C}^{n+1}/\mathcal{C}Q^n$ of \mathcal{C} relative to Q is Cohen-Macaulay and, by [10, Theorem 2.1], we have

$$e_1(\mathcal{C}) = \text{cdeg}(\mathbf{R}) + \sum_{j=1}^{\rho(\mathbf{R})-1} \lambda(\mathcal{C}^{j+1}/a\mathcal{C}^j) = \sum_{j=0}^{\rho(\mathbf{R})-1} \lambda(\mathcal{C}^{j+1}/a\mathcal{C}^j).$$

Remark 4.4. Let \mathbf{R} be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Then the multiplicity of the Sally module $s_0(S) = e_1(\mathcal{C}) - e_0(\mathcal{C}) + \lambda(\mathbf{R}/\mathcal{C}) = e_1(\mathcal{C}) - \text{cdeg}(\mathbf{R})$ is an invariant of the ring \mathbf{R} , by [11, Corollary 2.8] and Proposition 2.1.

The following property of Cohen-Macaulay rings of type 2 is a useful calculation that we will use to characterize rings with minimal canonical index.

Proposition 4.5. *Let \mathbf{R} be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Let (a) be a minimal reduction of \mathcal{C} . If $\nu(\mathcal{C}) = 2$, then $\lambda(\mathcal{C}^2/a\mathcal{C}) = \lambda(\mathcal{C}/(a))$.*

Proof. Let $\mathcal{C} = (a, b)$ and consider the exact sequence

$$0 \rightarrow Z \rightarrow \mathbf{R}^2 \rightarrow \mathcal{C} \rightarrow 0,$$

where $Z = \{(r, s) \in \mathbf{R}^2 \mid ra + sb = 0\}$. By tensoring this exact sequence with \mathbf{R}/\mathcal{C} , we obtain

$$Z/\mathcal{C}Z \xrightarrow{g} (\mathbf{R}/\mathcal{C})^2 \xrightarrow{h} \mathcal{C}/\mathcal{C}^2 \rightarrow 0.$$

Then we have

$\ker(h) = \text{Im}(g) \simeq (Z/\mathcal{C}Z)/((Z \cap \mathcal{C}\mathbf{R}^2)/\mathcal{C}Z) \simeq Z/(Z \cap \mathcal{C}\mathbf{R}^2) \simeq (Z/B)/((Z \cap \mathcal{C}\mathbf{R}^2)/B)$,
where $B = \{(-bx, ax) \mid x \in \mathbf{R}\}$.

We claim that $Z \cap \mathcal{C}\mathbf{R}^2 \subset B$, i.e., $\delta(\mathcal{C}) = (Z \cap \mathcal{C}\mathbf{R}^2)/B = 0$. Let $(r, s) \in Z \cap \mathcal{C}\mathbf{R}^2$. Then

$$ra + sb = 0 \Rightarrow \frac{s}{a} \cdot b = -r \in \mathcal{C} \text{ and } \frac{s}{a} \cdot a = s \in \mathcal{C}.$$

Denote the total ring of fractions of \mathbf{R} by K . Since \mathcal{C} is a canonical ideal, we have

$$\frac{s}{a} \in \mathcal{C} :_K \mathcal{C} = \mathbf{R}.$$

Therefore

$$(r, s) = \left(-b \cdot \frac{s}{a}, a \cdot \frac{s}{a}\right) \in B.$$

Hence $\ker(h) \simeq Z/B = H_1(\mathcal{C})$ and we obtain the following exact sequence

$$0 \rightarrow H_1(\mathcal{C}) \rightarrow (\mathbf{R}/\mathcal{C})^2 \rightarrow \mathcal{C}/\mathcal{C}^2 \rightarrow 0.$$

Next we claim that $\lambda(H_1(\mathcal{C})) = \lambda(\mathbf{R}/\mathcal{C})$. Note that $H_1(\mathcal{C}) \simeq ((a) : b)/(a)$ by mapping $(r, s) + B$ with $ra + sb = 0$ to $s + (a)$. Using the exact sequence

$$0 \rightarrow ((a) : b)/(a) \rightarrow \mathbf{R}/(a) \xrightarrow{\cdot b} \mathbf{R}/(a) \rightarrow \mathbf{R}/\mathcal{C} \rightarrow 0,$$

we get

$$\lambda(\mathbf{R}/\mathcal{C}) = \lambda(((a) : b)/(a)) = \lambda(H_1(\mathcal{C})).$$

Now, using the exact sequence

$$0 \rightarrow H_1(\mathcal{C}) \rightarrow (\mathbf{R}/\mathcal{C})^2 \rightarrow \mathcal{C}/\mathcal{C}^2 \rightarrow 0,$$

we get

$$\lambda(\mathcal{C}/\mathcal{C}^2) = 2\lambda(\mathbf{R}/\mathcal{C}) - \lambda(H_1(\mathcal{C})) = \lambda(\mathbf{R}/\mathcal{C}).$$

Hence,

$$\lambda(\mathcal{C}^2/a\mathcal{C}) = \lambda(\mathcal{C}/a\mathcal{C}) - \lambda(\mathcal{C}/\mathcal{C}^2) = \lambda(\mathcal{C}/a\mathcal{C}) - \lambda(\mathbf{R}/\mathcal{C}) = \lambda(\mathcal{C}/a\mathcal{C}) - \lambda((a)/a\mathcal{C}) = \lambda(\mathcal{C}/(a)).$$

□

Theorem 4.6. *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Suppose that the type of \mathbf{R} is 2. Then we have the following.*

- (1) $e_1(\mathcal{C}) \leq \rho(\mathbf{R}) \text{cdeg}(\mathbf{R})$.
- (2) $\rho(\mathbf{R}) = 2$ if and only if $e_1(\mathcal{C}) = 2 \text{cdeg}(\mathbf{R})$.

Proof. Let $\mathcal{C} = (a, b)$, where (a) is a minimal reduction of \mathcal{C} .

(1) For each $j = 0, \dots, \rho(\mathbf{R}) - 1$, the module $\mathcal{C}^{j+1}/a\mathcal{C}^j$ is cyclic and annihilated by $L = \text{ann}(\mathcal{C}/(a))$. Hence we obtain

$$e_1(\mathcal{C}) = \sum_{j=0}^{\rho(\mathbf{R})-1} \lambda(\mathcal{C}^{j+1}/a\mathcal{C}^j) \leq \rho(\mathbf{R}) \lambda(\mathbf{R}/L) = \rho(\mathbf{R}) \text{cdeg}(\mathbf{R}).$$

(2) Note that $\rho(\mathbf{R}) = 2$ if and only if $e_1(\mathcal{C}) = \sum_{j=0}^1 \lambda(\mathcal{C}^{j+1}/a\mathcal{C}^j)$. Since $\nu(\mathcal{C}) = r(\mathbf{R}) = 2$, by

Proposition 4.5, $\lambda(\mathcal{C}/(a)) = \lambda(\mathcal{C}^2/a\mathcal{C})$. Thus, the assertion follows from

$$\sum_{j=0}^1 \lambda(\mathcal{C}^{j+1}/a\mathcal{C}^j) = 2 \lambda(\mathcal{C}/(a)) = 2 \text{cdeg}(\mathbf{R}).$$

□

Example 4.7. Let $H = \langle a, b, c \rangle$ be a numerical semigroup which is minimally generated by positive integers a, b, c with $\gcd(a, b, c) = 1$. If the semigroup ring $\mathbf{R} = k[[t^a, t^b, t^c]]$ is not a Gorenstein ring, then $r(\mathbf{R}) = 2$ (see [11, Section 4]).

Example 4.8. Let $A = k[X, Y, Z]$, let $I = (X^2 - YZ, Y^2 - XZ, Z^2 - XY)$ and $\mathbf{R} = A/I$. Let x, y, z be the images of X, Y, Z in \mathbf{R} . By [16, Theorem 10.6.5], we see that $\mathcal{C} = (x, z)$ is a canonical ideal of \mathbf{R} with a minimal reduction (x) . It is easy to see that $\rho(\mathbf{R}) = 2$, $e_1(\mathcal{C}) = 2$ and $\text{cdeg}(\mathbf{R}) = 1$.

Example 4.9. Let $A = k[X, Y, Z]$, let $I = (X^4 - Y^2Z^2, Y^4 - X^2Z^2, Z^4 - X^2Y^2)$ and $\mathbf{R} = A/I$. Let x, y, z be the images of X, Y, Z in \mathbf{R} . Then $\mathcal{C} = (x^2, z^2)$ is a canonical ideal of \mathbf{R} with a minimal reduction (x^2) . We have that $\rho(\mathbf{R}) = 2$, $e_1(\mathcal{C}) = 16$ and $\text{cdeg}(\mathbf{R}) = 8$.

Lower and upper bounds for the canonical index.

Example 4.10. Let $e \geq 4$ be an integer and let $H = \langle e, \{e + i\}_{3 \leq i \leq e-1}, 3e + 1, 3e + 2 \rangle$. Let k be a field and $V = k[[t]]$ the formal power series ring over k . Consider the semigroup ring $\mathbf{R} = k[[H]] \subseteq V$.

- (1) The conductor of H is $c = 2e + 3$.
- (2) The canonical module is $K_{\mathbf{R}} = \langle 1, t \rangle$ and $K_{\mathbf{R}}^{e-2} \subsetneq K_{\mathbf{R}}^{e-1} = V$.
- (3) The canonical ideal of \mathbf{R} is $\mathcal{C} = (t^e K_{\mathbf{R}})$ and $Q = (t^c)$ is a minimal reduction of \mathcal{C} . Moreover, $\rho(\mathbf{R}) = \text{red}(\mathcal{C}) = e - 1$.
- (4) The canonical degree is $\text{cdeg}(\mathbf{R}) = \lambda(\mathcal{C}/Q) = \lambda(K_{\mathbf{R}}/\mathbf{R}) = 3$.
- (5) In particular, $\text{cdeg}(\mathbf{R}) \leq \rho(\mathbf{R})$.

Proposition 4.11. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field and a canonical ideal \mathcal{C} . Suppose that $\mathbf{R}_{\mathfrak{p}}$ is a Gorenstein ring for $\forall \mathfrak{p} \in \text{Spec}(\mathbf{R}) \setminus \{\mathfrak{m}\}$ and that \mathcal{C} is equimultiple.

- (1) \mathcal{C}^n has finite local cohomology for all $n > 0$.
- (2) Let $\mathbb{I}(\mathcal{C}^n)$ denote the Buchsbaum invariant of \mathcal{C}^n . Then the nonnegative integer $\beta(\mathbf{R}) = \sup_{n > 0} \mathbb{I}(\mathcal{C}^n)$ is independent of the choice of \mathcal{C} .
- (3) $\rho(\mathbf{R}) \leq \deg(\mathbf{R}) + \beta(\mathbf{R}) - 1$.

Proof. (1) The assertion follows from $\mathcal{C}^n \mathbf{R}_{\mathfrak{p}} = a^n \mathbf{R}_{\mathfrak{p}}$ where $Q = (a)$ is a reduction of \mathcal{C} .

(2) We have $\mathcal{C}^n \cong \mathcal{D}^n$ for any canonical ideal \mathcal{D} and $\mathcal{C}^{n+1} = a\mathcal{C}^n$ for $\forall n \gg 0$.

(3) Let \mathfrak{q} be a minimal reduction of \mathfrak{m} . Then, for $\forall n > 0$, we have

$$\nu(\mathcal{C}^n) = \lambda(\mathcal{C}^n/\mathfrak{m}\mathcal{C}^n) \leq \lambda(\mathcal{C}^n/\mathfrak{q}\mathcal{C}^n) \leq e_0(\mathfrak{q}, \mathcal{C}^n) + \mathbb{I}(\mathcal{C}^n) \leq \deg(\mathbf{R}) + \beta(\mathbf{R}).$$

The conclusion follows from [8, Theorem 1]. □

Example 4.12. Consider the following examples of 1-dimensional Cohen-Macaulay semigroup rings \mathbf{R} with a canonical ideal \mathcal{C} such that $\text{cdeg}(\mathbf{R}) = r(\mathbf{R})$.

- (1) Let $a \geq 4$ be an integer and let $H = \langle a, a + 3, \dots, 2a - 1, 2a + 1, 2a + 2 \rangle$. Let $\mathbf{R} = k[[H]]$. Then the canonical module of \mathbf{R} is $K_{\mathbf{R}} = \langle 1, t, t^3, t^4, \dots, t^{a-1} \rangle$. The ideal $Q = (t^{a+3})$ is a minimal reduction of $\mathcal{C} = (t^{a+3}, t^{a+4}, t^{a+6}, t^{a+7}, \dots, t^{2a+2})$,

$$\text{cdeg}(\mathbf{R}) = a - 1 = \nu(\mathcal{C}), \quad \text{and} \quad \text{red}(\mathcal{C}) = 2.$$

- (2) Let $a \geq 5$ be an integer and let $H = \langle a, a + 1, a + 4, \dots, 2a - 1, 2a + 2, 2a + 3 \rangle$. Let $\mathbf{R} = k[[H]]$. Then the canonical module of \mathbf{R} is $K_{\mathbf{R}} = \langle 1, t, t^4, t^5, \dots, t^{a-1} \rangle$.

The ideal $Q = (t^{a+4})$ is a minimal reduction of $\mathcal{C} = (t^{a+4}, t^{a+5}, t^{a+8}, t^{a+9}, \dots, t^{2a+3})$.

$$\text{cdeg}(\mathbf{R}) = a - 2 = \nu(\mathcal{C}), \quad \text{and} \quad \text{red}(\mathcal{C}) = 3.$$

5. ROOTS OF CANONICAL IDEALS

Another phenomenon concerning canonical ideals is the following that may impact the value of $\text{cdeg}(\mathbf{R})$.

Definition 5.1. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with a canonical ideal \mathcal{C} . An ideal L is called a *root* of \mathcal{C} if $L^n \simeq \mathcal{C}$ for some n . In this case, we write $\tau_L(\mathcal{C}) = \min\{n \mid L^n \simeq \mathcal{C}\}$. Then the *rootset* of \mathbf{R} is the set $\text{root}(\mathbf{R}) = \{\tau_L(\mathcal{C}) \mid L \text{ is a root of } \mathcal{C}\}$.

The terminology ‘roots’ of \mathcal{C} already appears in [3]. Here is a simple example.

Example 5.2. ([3, Example 3.4]) Let $\mathbf{R} = k[[t^4, t^5, t^6, t^7]]$. Then $\mathcal{C} = (t^4, t^5, t^6)$ is a canonical ideal of \mathbf{R} . Let $I = (t^4, t^5)$. Then $I^2 = t^4\mathcal{C}$, that is, I is a square root of \mathcal{C} .

The set $\text{root}(\mathbf{R})$ is clearly independent of the chosen canonical ideal. To make clear the character of this set we appeal to the following property. We use the terminology of [3] calling an ideal L of \mathbf{R} *closed* if $\text{Hom}(L, L) = \mathbf{R}$.

Proposition 5.3. *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Let L be a root of \mathcal{C} . If $\text{Hom}(L^n, L^n) \simeq \mathbf{R}$ for infinitely many values of n then all powers of L are closed. In this case L is invertible and \mathbf{R} is Gorenstein.*

Proof. A property of roots is that they are closed ideals. More generally, it is clear that if $\text{Hom}(L^m, L^m) = \mathbf{R}$ then $\text{Hom}(L^n, L^n) = \mathbf{R}$ for $n < m$, which shows the first assertion.

We may assume that \mathbf{R} has an infinite residue field. Let $s = \text{red}(L)$. Then $L^{s+1} = xL^s$, for a minimal reduction (x) and thus $L^{2s} = x^s L^s$, which gives that

$$x^{-s}L^s \subset \text{Hom}(L^s, L^s) \simeq \mathbf{R}.$$

It follows that $L^s \subset (x^s) \subset L^s$, and thus $L^s = (x^s)$. Now taking t such that $L^t \simeq \mathcal{C}$ shows that \mathcal{C} is principal. \square

Corollary 5.4. *If $L^m \simeq \mathcal{C} \simeq L^n$, for $m \neq n$, then \mathcal{C} is principal.*

Proof. Suppose $m > n$. Then

$$\mathcal{C} \simeq L^m = L^n L^{m-n} \simeq L^m L^{m-n} = L^{2m-n}.$$

Iterating, L is a root of \mathcal{C} of arbitrarily high order. \square

Corollary 5.5. ([3, Proposition 3.8]) *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . If \mathbf{R} is not Gorenstein then no proper power of \mathcal{C} is a canonical ideal.*

Proposition 5.6. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field and with an equimultiple canonical ideal \mathcal{C} . Let L be a root of \mathcal{C} . Then $\tau_L(\mathcal{C}) \leq \min\{r(\mathbf{R}) - 1, \text{red}(L)\}$.*

Proof. Suppose $n = \tau_L(\mathcal{C}) \geq r(\mathbf{R})$. Then

$$\nu(L^n) = \nu(\mathcal{C}) = r(\mathbf{R}) < n + 1 = \binom{n+1}{1}.$$

By [8, Theorem 1], there exists a reduction (a) of L such that $L^n = aL^{n-1}$. Thus, $L^{n-1} \simeq \mathcal{C}$, which contradicts the minimality of $\tau_L(\mathcal{C})$. \square

Remark 5.7. We have the following.

- (1) If $r(\mathbf{R}) = 2$, then the isomorphism class of \mathcal{C} is the only root.

- (2) The upper bound in Proposition 5.6 is sharp. For example, let $a \geq 3$ be an integer and we consider the numerical semigroup ring $\mathbf{R} = k[[\{t^i\}_{a \leq i \leq 2a-1}]] \subseteq k[[t]]$. Then the canonical module of \mathbf{R} is

$$K_{\mathbf{R}} = \sum_{i=0}^{a-2} \mathbf{R}t^i = (\mathbf{R} + \mathbf{R}t)^{a-2}.$$

Thus \mathbf{R} has a canonical ideal $\mathcal{C} = t^{a(a-2)}K_{\mathbf{R}}$. Let $L = (t^a, t^{a+1})$. Then $\mathcal{C} = L^{a-2}$. Hence we have $\tau_L(\mathcal{C}) = a - 2 = r(\mathbf{R}) - 1$.

From Proposition 5.6 we have the following.

Theorem 5.8. *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal. If \mathbf{R} is not Gorenstein, then $\text{root}(\mathbf{R})$ is a finite set of cardinality less than $r(\mathbf{R})$.*

Applications of roots.

Proposition 5.9. *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal \mathcal{C} . Let f be the supremum of the reduction numbers of the \mathfrak{m} -primary ideals. Suppose that $L^n \simeq \mathcal{C}$. If p divides n , then $\rho(\mathbf{R}) \leq (f + p - 1)/p$.*

Proof. Since $n = pm$, by replacing L^m by I , we may assume that $I^p = \mathcal{C}$. Let $r = \text{red}(I)$ with $I^{r+1} = xI^r$. Then $r = ps + q$ for some q such that $-p + 1 \leq q \leq 0$. Since $ps = r - q \geq r$, we have

$$\mathcal{C}^{s+1} = I^{ps+p} = x^p I^{ps} = x^p \mathcal{C}^s.$$

Thus, $\rho(\mathbf{R}) = \text{red}(\mathcal{C}) \leq s = (r - q)/p \leq (r + p - 1)/p \leq (f + p - 1)/p$. \square

A computation of roots of the canonical ideal. Let $0 < a_1 < a_2 < \dots < a_q$ be integers such that $\gcd(a_1, a_2, \dots, a_q) = 1$. Let $H = \langle a_1, a_2, \dots, a_q \rangle$ be the numerical semigroup generated by a_i 's. Let $V = k[[t]]$ be the formal power series ring over a field k and set $\mathbf{R} = k[[t^{a_1}, t^{a_2}, \dots, t^{a_q}]]$. We denote by \mathfrak{m} the maximal ideal of \mathbf{R} and by $e = a_1$ the multiplicity of \mathbf{R} . Let v be the discrete valuation of V . In what follows, let $\mathbf{R} \subseteq L \subseteq V$ be a finitely generated \mathbf{R} -submodule of V such that $\nu(L) > 1$. We set $\ell = \nu(L) - 1$. Then we have the following.

Lemma 5.10. *With notation as above, $1 \notin \mathfrak{m}L$.*

Proof. Choose $0 \neq g \in \mathfrak{m}$ so that $gV \subsetneq \mathbf{R}$. Then $Q = g\mathbf{R}$ is a minimal reduction of the \mathfrak{m} -primary ideal $I = gL$ of \mathbf{R} , so that $g \notin \mathfrak{m}I$. Hence $1 \notin \mathfrak{m}L$. \square

Lemma 5.11. *With notation as above, there exist elements $f_1, f_2, \dots, f_\ell \in L$ such that*

- (1) $L = \mathbf{R} + \sum_{i=1}^{\ell} \mathbf{R}f_i$,
- (2) $0 < v(f_1) < v(f_2) < \dots < v(f_\ell)$, and
- (3) $v(f_i) \notin H$ for all $1 \leq i \leq \ell$.

Proof. Let $L = \mathbf{R} + \sum_{i=1}^{\ell} \mathbf{R}f_i$ with $f_i \in L$. Let $1 \leq i \leq \ell$ and assume that $m = v(f_i) \in H$. We write $f_i = \sum_{j=m}^{\infty} c_j t^j$ with $c_j \in k$. Then $c_s \neq 0$ for some $s > m$ such that $s \notin H$, because $f_i \notin \mathbf{R}$. Choose such integer s as small as possible and set $h = f_i - \sum_{j=m}^{s-1} c_j t^j$. Then $\sum_{j=m}^{s-1} c_j t^j \in \mathbf{R}$ and $\mathbf{R} + \mathbf{R}f_i = \mathbf{R} + \mathbf{R}h$. Consequently, as $v(h) = s > m = v(f_i)$, replacing f_i with h , we may assume that $v(f_i) \notin H$ for all $1 \leq i \leq \ell$. Let $1 \leq i < j \leq \ell$ and assume that $v(f_i) = v(f_j) = m$. Then, since $f_j = cf_i + h$ for some $0 \neq c \in k$ and $h \in L$ such that $v(h) > m$, replacing f_j with h , we may assume that $v(f_j) > v(f_i)$. Therefore we can choose a minimal system of generators of L satisfying conditions (2) and (3). \square

Lemma 5.12. *Let $f_0, f_1, f_2, \dots, f_\ell \in V$. Assume that $L = \langle f_0, f_1, f_2, \dots, f_\ell \rangle$ and $v(f_0) < v(f_1) < v(f_2) < \dots < v(f_\ell)$. Then $L = \langle 1, f_1, f_2, \dots, f_\ell \rangle$.*

Proof. Since $1 \in L$ and $L \subseteq V$, we have $v(f_0) = 0$. We may assume that $f_0 = 1 + \xi$ with $\xi \in tV$. Then $\xi \in L$ as $1 \in L$. We write $\xi = \alpha_0 f_0 + \alpha_1 f_1 + \dots + \alpha_\ell f_\ell$ with $\alpha_i \in \mathbf{R}$. Then $v(\alpha_0) = v(\alpha_0 f_0) > 0$ since $f_i \in tV$ for all $1 \leq i \leq \ell$, so that $\alpha_0 \in \mathfrak{m}$. Consequently $L/\mathfrak{m}L$ is spanned by the images of $1, \{f_i\}_{1 \leq i \leq \ell}$, whence $L = \langle 1, f_1, f_2, \dots, f_\ell \rangle$ as claimed. \square

Proposition 5.13. *With notation as above, let $1, f_1, f_2, \dots, f_\ell \in L$ and $1, g_1, g_2, \dots, g_\ell \in L$ be systems of generators of L and assume that both of them satisfy condition (2) in Lemma 5.11. Suppose that $v(f_\ell) < e = a_1$. Then $v(f_i) = v(g_i)$ for all $1 \leq i \leq \ell$.*

Proof. We set $m_i = v(f_i)$ and $n_i = v(g_i)$ for each $1 \leq i \leq \ell$. Let us write

$$\begin{aligned} f_1 &= \alpha_0 + \alpha_1 g_1 + \dots + \alpha_\ell g_\ell \\ g_1 &= \beta_0 + \beta_1 f_1 + \dots + \beta_\ell f_\ell \end{aligned}$$

with $\alpha_i, \beta_i \in \mathbf{R}$. Then $v(\beta_1 f_1 + \dots + \beta_\ell f_\ell) \geq v(f_1) = m_1 > 0$, whence $\beta_0 \in \mathfrak{m}$ because $v(g_1) = n_1 > 0$. We similarly have that $\alpha_0 \in \mathfrak{m}$. Therefore $n_1 = v(g_1) \geq m_1$, since $v(\beta_0) \geq e > m_\ell \geq m_1$ and $v(\beta_1 f_1 + \dots + \beta_\ell f_\ell) \geq m_1$. Suppose that $n_1 > m_1$. Then $v(\alpha_1 g_1 + \dots + \alpha_\ell g_\ell) \geq n_1 > m_1$ and $v(\alpha_0) \geq e > m_1$, whence $v(f_1) > m_1$, a contradiction. Thus $m_1 = n_1$.

Now let $1 \leq i < \ell$ and assume that $m_j = n_j$ for all $1 \leq j \leq i$. We want to show $m_{i+1} = n_{i+1}$. Let us write

$$\begin{aligned} f_{i+1} &= \gamma_0 + \gamma_1 g_1 + \dots + \gamma_\ell g_\ell \\ g_{i+1} &= \delta_0 + \delta_1 f_1 + \dots + \delta_\ell f_\ell \end{aligned}$$

with $\gamma_i, \delta_i \in \mathbf{R}$.

First we claim that $\gamma_j, \delta_j \in \mathfrak{m}$ for all $0 \leq j \leq i$. As above, we get $\gamma_0, \delta_0 \in \mathfrak{m}$. Let $0 \leq k < i$ and assume that $\gamma_j, \delta_j \in \mathfrak{m}$ for all $0 \leq j \leq k$. We show $\gamma_{k+1}, \delta_{k+1} \in \mathfrak{m}$. Suppose that $\delta_{k+1} \notin \mathfrak{m}$. Then as $v(\delta_0 + \delta_1 f_1 + \dots + \delta_k f_k) \geq e > m_{k+1}$, we get $v(\delta_0 + \delta_1 f_1 + \dots + \delta_k f_k + \delta_{k+1} f_{k+1}) = m_{k+1}$, so that $n_{i+1} = v(g_{i+1}) = v(\delta_0 + \delta_1 f_1 + \dots + \delta_\ell f_\ell) = m_{k+1}$, since $v(f_h) = m_h > m_{k+1}$ if $h > k+1$. This is impossible, since $n_{i+1} > n_i = m_i \geq m_{k+1}$. Thus $\delta_{k+1} \in \mathfrak{m}$. We similarly get $\gamma_{k+1} \in \mathfrak{m}$, and the claim is proved.

Consequently, since $v(\delta_0 + \delta_1 f_1 + \dots + \delta_i f_i) \geq e > m_{i+1}$ and $v(f_h) \geq m_{i+1}$ if $h \geq i+1$, we have $n_{i+1} = v(g_{i+1}) = v(\delta_0 + \delta_1 f_1 + \dots + \delta_\ell f_\ell) \geq m_{i+1}$. Assume that $n_{i+1} > m_{i+1}$. Then since $v(\gamma_0 + \gamma_1 g_1 + \dots + \gamma_i g_i) \geq e > m_{i+1}$ and $v(g_h) \geq n_{i+1} > m_{i+1}$ if $h \geq i+1$, we have $m_{i+1} = v(f_{i+1}) = v(\gamma_0 + \gamma_1 g_1 + \dots + \gamma_\ell g_\ell) > m_{i+1}$. This is a contradiction. Hence $m_{i+1} = n_{i+1}$, as desired. \square

Theorem 5.14. *With notation as above, assume that $r(\mathbf{R}) = 3$ and write the canonical module $K_{\mathbf{R}} = \langle 1, t^a, t^b \rangle$ with $0 < a < b$. Suppose that $b \neq 2a$ and $b < e$. Let J be an ideal of \mathbf{R} and let $n \geq 1$ be an integer. If $J^n \cong K_{\mathbf{R}}$, then $n = 1$.*

Proof. We already know $n \leq 2$ by Proposition 5.6. Suppose that $n = 2$. Let $f \in J$ such that $JV = fV$ and set $L = f^{-1}J$. Then $\mathbf{R} \subseteq L \subseteq V$ and $L^2 = f^{-2}J^2 \cong K_{\mathbf{R}}$. By Lemma 5.11, we can write $L = \langle 1, f_1, f_2, \dots, f_\ell \rangle$, where $\ell = \nu(L) - 1 \geq 1$, $0 < v(f_1) < v(f_2) < \dots < v(f_\ell)$, and $v(f_i) \notin H$ for all $1 \leq i \leq \ell$. Then $L^2 = \langle 1, \{f_i\}_{1 \leq i \leq \ell}, \{f_i f_j\}_{1 \leq i < j \leq \ell} \rangle$. Let $n_i = v(f_i)$ for each $1 \leq i \leq \ell$.

Claim 5.15. *Suppose that $\ell > 1$. Then $f_1 \notin \langle 1, \{f_i\}_{2 \leq i \leq \ell}, \{f_i f_j\}_{1 \leq i < j \leq \ell} \rangle$.*

Proof of Claim 5.15. Assume the contrary and write

$$f_1 = \alpha + \sum_{i=2}^{\ell} \alpha_i f_i + \sum_{1 \leq i < j \leq \ell} \alpha_{ij} f_i f_j$$

with $\alpha, \alpha_i, \alpha_{ij} \in \mathbf{R}$. Then since $n_1 < n_i$ for $i \geq 2$ and $n_1 \leq n_i < n_i + n_j$ for $1 \leq i < j \leq \ell$, we have $n_1 = v(\alpha)$, which is impossible, because $n_1 \notin H$ but $v(\alpha) \in H$. \square

Since $\nu(L^2) = 3$, we have $L^2 = \langle 1, f_1, f_i f_j \rangle$ for some $1 \leq i \leq j \leq \ell$. This is clear if $\ell = 1$, and it follows by Claim 5.15 if $\ell > 1$. In fact, the other possibility is $L^2 = \langle 1, f_1, f_i \rangle$ with $i \geq 2$. When this is the case, we get $L = L^2$, and so $\text{red}_{(f)} J \leq 1$, whence $\text{red}_{(f^2)} J^2 \leq 1$. Therefore, since $J^2 \cong K_{\mathbf{R}}$, \mathbf{R} is a Gorenstein ring, which is a contradiction.

We now choose $0 \neq \theta \in Q(V)$, where $Q(V)$ is the quotient field of V , so that $K_{\mathbf{R}} = \theta L^2$. Notice that θ is a unit of V (as $\mathbf{R} \subseteq L$ and $\mathbf{R} \subseteq K_{\mathbf{R}}$). Compare

$$K_{\mathbf{R}} = \langle 1, t^a, t^b \rangle = \langle \theta, \theta f_1, \theta f_i f_j \rangle$$

and notice that $\langle \theta, \theta f_1, \theta f_i f_j \rangle = \langle 1, \theta f_1, \theta f_i f_j \rangle$ by Lemma 5.12. Therefore since $0 < a < b < e$ and $0 < n_1 < n_i + n_j$, by Proposition 5.13 $n_1 = a$ and $n_i + n_j = b$; hence $b \geq 2a$. Furthermore, since $b \neq 2a$, we have $\ell > 1$.

Since $L \subseteq L^2$, we have $f_2 \in L^2 = \langle 1, f_1, f_i f_j \rangle$. Let us write

$$f_2 = \alpha + \beta f_1 + \gamma f_i f_j$$

with $\alpha, \beta, \gamma \in \mathbf{R}$. Then $\alpha \in \mathfrak{m}$, because $v(f_2) = n_2 > n_1$ and $v(\beta f_1 + \gamma f_i f_j) \geq n_1$. Hence $v(\alpha) > b$, because $e > b$. Consequently $\beta \in \mathfrak{m}$; otherwise $n_2 = v(f_2) = v(\beta f_1) = n_1$ (notice that $v(\alpha + \gamma f_i f_j) \geq b \geq 2n_1$). Therefore $v(\beta f_1) > b + n_1$, so that $n_2 = v(f_2) \geq b$. Since $b = n_i + n_j > n_j$, we then have $i = j = 1$. This is contradiction, since $b \neq 2a$. \square

Let us give examples satisfying the conditions stated in Theorem 5.14.

Example 5.16. Let $e \geq 7$ be an integer and set

$$H = \langle e + i \mid 0 \leq i \leq e - 2 \text{ such that } i \neq e - 4, e - 3 \rangle.$$

Then $K_{\mathbf{R}} = \langle 1, t^2, t^3 \rangle$ and $r(\mathbf{R}) = 3$. More generally, let $a, b, e \in \mathbb{Z}$ such that $0 < a < b$, $b < 2a$, and $e \geq a + b + 2$. We consider the numerical semigroup

$$H = \langle e + i \mid 0 \leq i \leq e - 2 \text{ such that } i \neq e - b - 1, e - a - 1 \rangle.$$

Then $K_{\mathbf{R}} = \langle 1, t^a, t^b \rangle$ and $r(\mathbf{R}) = 3$. These rings \mathbf{R} contain no ideals L such that $L^n \cong K_{\mathbf{R}}$ for some integer $n \geq 2$.

6. CHANGE OF RINGS

Degree formulas are often statements about change of rings. Let us see how this may work for $\text{cdeg}(\mathbf{R})$. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with a canonical ideal \mathcal{C} . Let $(\mathbf{A}, \mathfrak{n})$ be a Cohen-Macaulay local ring with a finite injective morphism $\varphi : \mathbf{R} \rightarrow \mathbf{A}$. Suppose that the total ring of fractions of \mathbf{A} is Gorenstein. Then a canonical module of \mathbf{A} is $\mathcal{D} = \text{Hom}_{\mathbf{R}}(\mathbf{A}, \mathcal{C})$, which is isomorphic to an ideal of \mathbf{A} .

Example 6.1. Let $\mathbf{A} = K[x_1, \dots, x_d]^{(n)}$ be the n -Veronese subring of the polynomial ring in d variables. Fix the Noether normalization $\mathbf{R} = K[x_1^n, \dots, x_d^n]$. The canonical module of \mathbf{A} is $\text{Hom}_{\mathbf{R}}(\mathbf{A}, \mathbf{R})$.

We want to develop relationships between the pairs $\{\text{cdeg}(\mathbf{R}), \text{red}(\mathcal{C})\}$ and $\{\text{cdeg}(\mathbf{A}), \text{red}(\mathcal{D})\}$.

Polynomial extension. Let us compare $\text{cdeg}(\mathbf{R})$ to $\text{cdeg}(\mathbf{R}[x]_{\mathfrak{m}_{\mathbf{R}}[x]})$ and $\text{cdeg}(\mathbf{R}[[x]])$.

Lemma 6.2. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with a canonical ideal \mathcal{C} . Let $\varphi : \mathbf{R} \rightarrow \mathbf{A}$ be a flat injective morphism, where $\mathbf{A} = \mathbf{R}[x]_{\mathfrak{m}_{\mathbf{R}}[x]}$ or $\mathbf{A} = \mathbf{R}[[x]]$.

- (1) The morphism φ has Gorenstein fiber.
- (2) $\mathcal{C} \otimes \mathbf{A}$ is a canonical ideal of \mathbf{A} .
- (3) If \mathcal{C} is equimultiple, then for the minimal reduction (a) of \mathcal{C} ,

$$\deg(\mathcal{C}/(a)) = \deg(\mathcal{C}/(a) \otimes \mathbf{A}).$$

Proof. Recall that if $\mathbf{A} = \mathbf{R}[x]_{\mathfrak{m}\mathbf{R}[x]}$, then $\mathbf{A}/\mathfrak{m}\mathbf{A} = k(x)$, and if $\mathbf{A} = \mathbf{R}[[x]]$, then $\mathbf{A}/\mathfrak{m}\mathbf{A} = k[[x]]$. This proves (1). Notice that the embedding $\mathcal{C} \subset \mathbf{R}$ leads to the embedding $\mathcal{C} \otimes \mathbf{A} \subset \mathbf{A}$. Assertion (2) follows from [1, Theorem 4.1], or [14, Satz 6.14]. \square

Proposition 6.3. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with a canonical ideal \mathcal{C} . Then $\text{cdeg}(\mathbf{R}) = \text{cdeg}(\mathbf{R}[x]_{\mathfrak{m}\mathbf{R}[x]})$.*

Proof. Since $\text{Min}(\mathcal{C} \otimes \mathbf{A}) = \{\mathfrak{p}\mathbf{A} \mid \mathfrak{p} \in \text{Min}(\mathcal{C})\}$ and $\deg(\mathbf{R}/\mathfrak{p}) = \deg(\mathbf{A}/\mathfrak{p}\mathbf{A})$, $\text{cdeg}(\mathbf{R})$ and $\text{cdeg}(\mathbf{A})$ are defined by the same summations. \square

Question 6.4. Is it true that $\text{cdeg}(\mathbf{R}) = \text{cdeg}(\mathbf{R}[[x]])$? If \mathcal{C} is equimultiple, then it is true. What if \mathcal{C} is not equimultiple?

Completion.

Proposition 6.5. *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with infinite residue field and a canonical ideal. Then $\text{cdeg}(\widehat{\mathbf{R}}) = \text{cdeg}(\mathbf{R})$.*

Proof. Let (a) be a minimal reduction of the canonical ideal \mathcal{C} . Then $a\widehat{\mathbf{R}}$ is a minimal reduction of the canonical ideal $\mathcal{C}\widehat{\mathbf{R}}$ of $\widehat{\mathbf{R}}$. Thus, we have

$$\text{cdeg}(\widehat{\mathbf{R}}) = \lambda_{\widehat{\mathbf{R}}}(\mathcal{C}\widehat{\mathbf{R}}/a\widehat{\mathbf{R}}) = \lambda_{\widehat{\mathbf{R}}}(\widehat{\mathbf{R}} \otimes_{\mathbf{R}} \mathcal{C}/(a)) = \lambda_{\mathbf{R}}(\mathcal{C}/(a))\lambda_{\widehat{\mathbf{R}}}(\widehat{\mathbf{R}}/\mathfrak{m}\widehat{\mathbf{R}}) = \text{cdeg}(\mathbf{R}).$$

\square

Theorem 6.6. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field and a canonical ideal. Then we have the following.*

- (1) $\text{cdeg}(\widehat{\mathbf{R}}) \geq \text{cdeg}(\mathbf{R})$.
- (2) $\text{cdeg}(\widehat{\mathbf{R}}) = \text{cdeg}(\mathbf{R})$ if and only if for every $\mathfrak{p} \in \text{Spec}(\mathbf{R})$ with $\dim \mathbf{R}_{\mathfrak{p}} = 1$, $\mathbf{R}_{\mathfrak{p}}$ is a Gorenstein ring, or $\mathfrak{p}\widehat{\mathbf{R}} \in \text{Spec}(\widehat{\mathbf{R}})$.

Proof. Let \mathcal{C} be a canonical ideal of \mathbf{R} . We set $X = \text{Ass}_{\mathbf{R}} \mathbf{R}/\mathcal{C}$ and $Y = \text{Ass}_{\widehat{\mathbf{R}}} \widehat{\mathbf{R}}/\mathcal{C}\widehat{\mathbf{R}}$. Then

$$Y = \bigcup_{\mathfrak{p} \in X} \text{Ass}_{\widehat{\mathbf{R}}} \widehat{\mathbf{R}}/\mathfrak{p}\widehat{\mathbf{R}}.$$

Notice that for each $\mathfrak{p} \in X$, $Z = \text{Ass}_{\widehat{\mathbf{R}}} \widehat{\mathbf{R}}/\mathfrak{p}\widehat{\mathbf{R}} = \{P \in Y \mid P \cap \mathbf{R} = \mathfrak{p}\}$.

We now fix an element $\mathfrak{p} \in X$ and let $S = \widehat{\mathbf{R}} \setminus \bigcup_{P \in Z} P$. We set $A = \mathbf{R}_{\mathfrak{p}}$ and $B = S^{-1}\widehat{\mathbf{R}}$. Then we have a flat homomorphism

$$A = \mathbf{R}_{\mathfrak{p}} \rightarrow \widehat{\mathbf{R}}_{\mathfrak{p}} \rightarrow B = S^{-1}\widehat{\mathbf{R}},$$

since the homomorphism $\widehat{\mathbf{R}}_{\mathfrak{p}} \rightarrow B$ is a localization (by the image of S under the map $\widehat{\mathbf{R}} \rightarrow \widehat{\mathbf{R}}_{\mathfrak{p}}$). We choose an element $a \in \mathcal{C}$ so that $a\mathbf{R}_{\mathfrak{p}}$ is a reduction of $\mathcal{C}\mathbf{R}_{\mathfrak{p}}$. We then have that

$$\lambda_B(B \otimes_A X) = \lambda_A(X) \cdot \lambda_B(B/\mathfrak{p}B)$$

for every A -module X with $\lambda_A(X) < \infty$ and that

$$\lambda_B(B/\mathfrak{p}B) = \sum_{P \in Z} \lambda_{\widehat{\mathbf{R}}_P}(\widehat{\mathbf{R}}_P/\mathfrak{p}\widehat{\mathbf{R}}_P).$$

Hence

$$\sum_{P \in Z} \lambda_{\widehat{\mathbf{R}}_P}(\mathcal{C}\widehat{\mathbf{R}}_P/a\widehat{\mathbf{R}}_P) = \lambda_B(\mathcal{C}B/aB) = \lambda_B(B/\mathfrak{p}B) \cdot \lambda_A(\mathcal{C}A/aA).$$

Consequently

$$\begin{aligned} \text{cdeg}(\widehat{\mathbf{R}}) &= \sum_{\mathfrak{p} \in X} \left(\sum_{P \in Z} \lambda_{\widehat{\mathbf{R}}_P}(\mathcal{C}\widehat{\mathbf{R}}_P/a\widehat{\mathbf{R}}_P) \right) = \sum_{\mathfrak{p} \in X} \lambda_B(B/\mathfrak{p}B) \cdot \lambda_A(\mathcal{C}A/aA) \\ &\geq \sum_{\mathfrak{p} \in X} \lambda_{\mathbf{R}_P}(\mathcal{C}\mathbf{R}_P/a\mathbf{R}_P) = \text{cdeg}(\mathbf{R}), \end{aligned}$$

which shows assertion (1).

We have

$$\begin{aligned} \text{cdeg}(\widehat{\mathbf{R}}) = \text{cdeg}(\mathbf{R}) &\Leftrightarrow \forall \mathfrak{p} \in X = \text{Ass}_{\mathbf{R}}(\mathbf{R}/\mathcal{C}), \mathbf{R}_{\mathfrak{p}} \text{ is a Gorenstein ring, or } \lambda_B(B/\mathfrak{p}B) = 1 \\ &\Leftrightarrow \forall \mathfrak{p} \in X, \mathbf{R}_{\mathfrak{p}} \text{ is a Gorenstein ring, or } \text{Ass}_{\widehat{\mathbf{R}}} \widehat{\mathbf{R}}/\mathfrak{p}\widehat{\mathbf{R}} \\ &\quad \text{is a singleton and } \widehat{\mathbf{R}}_P/\mathfrak{p}\widehat{\mathbf{R}}_P \text{ is a field where } \{P\} = \text{Ass}_{\widehat{\mathbf{R}}} \widehat{\mathbf{R}}/\mathfrak{p}\widehat{\mathbf{R}} \\ &\Leftrightarrow \forall \mathfrak{p} \in X, \mathbf{R}_{\mathfrak{p}} \text{ is a Gorenstein ring, or } \mathfrak{p}\widehat{\mathbf{R}} \in \text{Spec}(\widehat{\mathbf{R}}). \end{aligned}$$

Assume $\text{cdeg}(\widehat{\mathbf{R}}) = \text{cdeg}(\mathbf{R})$. Let $\mathfrak{p} \in \text{Spec}(\mathbf{R})$ with $\dim \mathbf{R}_{\mathfrak{p}} = 1$ and take a non-zerodivisor $f \in \mathfrak{p}$. Then since $f\mathcal{C} \cong \mathcal{C}$ and $\mathfrak{p} \in \text{Ass}_{\mathbf{R}} \mathbf{R}/f\mathcal{C}$, $\mathbf{R}_{\mathfrak{p}}$ is a Gorenstein ring or $\mathfrak{p}\widehat{\mathbf{R}} \in \text{Spec}(\widehat{\mathbf{R}})$. Thus assertion (2) follows. \square

Augmented ring. Motivated by [11, Theorem 6.5] let us determine the canonical degree of a class of local rings.

Let \mathbf{R} be a commutative ring with total quotient ring $\mathbf{Q}(\mathbf{R})$ and let \mathcal{F} denote the set of \mathbf{R} -submodules of $\mathbf{Q}(\mathbf{R})$. Let $M, K \in \mathcal{F}$. Let $M^\vee = \text{Hom}_{\mathbf{R}}(M, K)$ and let $\mathbf{A} = \mathbf{R} \ltimes M$ denote the idealization of M over \mathbf{R} . Then the \mathbf{R} -module $M^\vee \oplus K$ becomes an \mathbf{A} -module under the action

$$(a, m) \circ (f, x) = (af, f(m) + ax),$$

where $(a, m) \in \mathbf{A}$ and $(f, x) \in M^\vee \times K$. We notice that the canonical homomorphism $\varphi : \text{Hom}_{\mathbf{R}}(\mathbf{A}, K) \rightarrow M^\vee \times K$ such that $\varphi(f) = (f \circ \lambda, f(1))$ is an \mathbf{A} -isomorphism, where $\lambda : M \rightarrow \mathbf{A}, \lambda(m) = (0, m)$. We also notice that $K : M \in \mathcal{F}$ and $(K : M) \times K \subseteq \mathbf{Q}(\mathbf{R}) \ltimes \mathbf{Q}(\mathbf{R})$ is an \mathbf{A} -submodule of $\mathbf{Q}(\mathbf{R}) \ltimes \mathbf{Q}(\mathbf{R})$, the idealization of $\mathbf{Q}(\mathbf{R})$ over itself. When $\mathbf{Q}(\mathbf{R}) \cdot M = \mathbf{Q}(\mathbf{R})$, identifying $\text{Hom}_{\mathbf{R}}(M, K)$ with $K : M$, we have a natural isomorphism of \mathbf{A} -modules

$$\text{Hom}_{\mathbf{R}}(\mathbf{A}, K) \cong (K : M) \times K.$$

Proposition 6.7. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module $\mathbf{K}_{\mathbf{R}}$. Let M, K be \mathbf{R} -submodules of $\mathbf{Q}(\mathbf{R})$. Assume that M is a finitely generated \mathbf{R} -module with $\mathbf{Q}(\mathbf{R}) \cdot M = \mathbf{Q}(\mathbf{R})$ and that $K \cong \mathbf{K}_{\mathbf{R}}$ as an \mathbf{R} -module. Let $\mathbf{A} = \mathbf{R} \ltimes M$, and let $L = K : M$. Then $\mathbf{K}_{\mathbf{A}} = (K : M) \times K$ in $\mathbf{Q}(\mathbf{R}) \ltimes \mathbf{Q}(\mathbf{R})$ is a canonical module of \mathbf{A} , and $(\mathbf{K}_{\mathbf{A}})^n = L^n \times L^{n-1}K$ for all $n \geq 1$.*

Proof. The assertions follow from the above observations. The proof of the equality $(\mathbf{K}_{\mathbf{A}})^n = L^n \times L^{n-1}K$ follows by induction on n . \square

Theorem 6.8. *Let $(\mathbf{R}, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with infinite residue field and a canonical ideal \mathcal{C} . Suppose \mathbf{R} is not a DVR. Then we have the following:*

$$\text{cdeg}(\mathbf{R} \ltimes \mathfrak{m}) = 2 \text{cdeg}(\mathbf{R}) + 2 \quad \text{and} \quad r(\mathbf{R} \ltimes \mathfrak{m}) = 2r(\mathbf{R}) + 1.$$

Proof. We may assume that $\mathcal{C} \subset \mathfrak{m}^2$ by replacing \mathcal{C} with $b\mathcal{C}$ if necessary, where $b \in \mathfrak{m}^2$ is a non-zerodivisor of \mathbf{R} . Let $\mathbf{A} = \mathbf{R} \ltimes \mathfrak{m}$ and set $L = \mathcal{C} : \mathfrak{m}$. Then by Proposition 6.7 $\mathcal{D} = L \times \mathcal{C}$ is a canonical ideal of \mathbf{A} (notice that $L \subset \mathfrak{m}$, since $\mathcal{C} \subset \mathfrak{m}^2$). Let $a \in \mathcal{C}$ and assume that (a) is a reduction of \mathcal{C} . Then since $L^2 = \mathcal{C}L$ by [6, Lemma 3.6 (a)] (notice that $\nu(\mathfrak{m}/\mathcal{C}) \geq 2$, since \mathbf{R}

is not a DVR), the ideal (a) is also a reduction of L , so that $a\mathbf{A}$ is a minimal reduction of the canonical ideal \mathcal{D} . Hence

$$\text{cdeg}(\mathbf{A}) = \lambda(\mathcal{D}/a\mathbf{A}) = \lambda(L/a\mathbf{R}) + \lambda(\mathcal{C}/a\mathbf{m}).$$

Because $\lambda(L/\mathcal{C}) = \lambda(\mathbf{R}/\mathbf{m}) = 1$, we get

$$\lambda(L/a\mathbf{R}) + \lambda(\mathcal{C}/a\mathbf{m}) = [\lambda(L/\mathcal{C}) + \lambda(\mathcal{C}/a\mathbf{R})] + [\lambda(\mathcal{C}/a\mathbf{R}) + \lambda(a\mathbf{R}/a\mathbf{m})] = 2\lambda(\mathcal{C}/a\mathbf{R}) + 2.$$

Thus $\text{cdeg}(\mathbf{A}) = 2\text{cdeg}(\mathbf{R}) + 2$. Notice that $\mathbf{n} = \mathbf{m} \times \mathbf{m}$ is the maximal of \mathbf{A} . Since $\lambda(L/\mathcal{C}) = 1$ and $\mathbf{m}\mathcal{C} = \mathbf{m}L$ by [6, Lemma 3.6 (b)], we have

$$\begin{aligned} r(\mathbf{A}) = \lambda(\mathcal{D}/\mathbf{n}\mathcal{D}) = \lambda((\mathcal{C}/\mathbf{m}\mathcal{C}) \oplus (L/\mathbf{m}L)) &= \lambda(\mathcal{C}/\mathbf{m}\mathcal{C}) + \lambda(L/\mathbf{m}\mathcal{C}) \\ &= \lambda(\mathcal{C}/\mathbf{m}\mathcal{C}) + [\lambda(L/\mathcal{C}) + \lambda(\mathcal{C}/\mathbf{m}\mathcal{C})] = 2r(\mathbf{R}) + 1 \end{aligned}$$

as claimed. \square

Corollary 6.9. ([11, Theorem 6.5]) *Let (\mathbf{R}, \mathbf{m}) be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal. Then \mathbf{R} is an almost Gorenstein ring if and only if $\mathbf{R} \times \mathbf{m}$ is an almost Gorenstein ring.*

Proof. It follows from Theorem 6.8 and Proposition 3.3. \square

Question 6.10. Let (\mathbf{R}, \mathbf{m}) be a 1-dimensional Cohen-Macaulay local ring with a canonical ideal. (i) Is it true that $\rho(\mathbf{R}) = \rho(\mathbf{R} \times \mathbf{m})$? (ii) Are the roots of $\mathbf{R} \times \mathbf{m}$ related to the roots of \mathbf{R} ?

Hyperplane sections. A change of rings issue is the comparison $\text{cdeg}(\mathbf{R})$ to $\text{cdeg}(\mathbf{R}/(x))$ for an appropriate regular element x . We know that if \mathcal{C} is a canonical module for \mathbf{R} then $\mathcal{C}/x\mathcal{C}$ is a canonical module for $\mathbf{R}/(x)$ with the same number of generators, so type is preserved under specialization. However $\mathcal{C}/x\mathcal{C}$ may not be isomorphic to an ideal of $\mathbf{R}/(x)$. Here is a case of good behavior. Suppose x is regular modulo \mathcal{C} . Then for the sequence

$$0 \rightarrow \mathcal{C} \rightarrow \mathbf{R} \rightarrow \mathbf{R}/\mathcal{C} \rightarrow 0,$$

we get the exact sequence

$$0 \rightarrow \mathcal{C}/x\mathcal{C} \rightarrow \mathbf{R}/(x) \rightarrow \mathbf{R}/(\mathcal{C}, x) \rightarrow 0,$$

so the canonical module $\mathcal{C}/x\mathcal{C}$ embeds in $\mathbf{R}/(x)$. Note that this leads $\text{red}(\mathcal{C}) \geq \text{red}(\mathcal{C}/x\mathcal{C})$.

Proposition 6.11. *Suppose (\mathbf{R}, \mathbf{m}) is a Cohen-Macaulay local ring of dimension $d \geq 2$ that has a canonical ideal \mathcal{C} . Suppose that \mathcal{C} is equimultiple and x is regular modulo \mathcal{C} . Then*

$$\text{cdeg}(\mathbf{R}) \leq \text{cdeg}(\mathbf{R}/(x)).$$

Proof. Let (a) be a minimal reduction of \mathcal{C} . Then its image in $\mathbf{R}/(x)$ is a minimal reduction of $\mathcal{C}/x\mathcal{C}$. Since x does not belong to any minimal primes of \mathcal{C} , which are the same as the minimal primes of (a) , the sequence a, x is a regular sequence of \mathbf{R} . Therefore, so is x, a . Thus by [9, Theorem 3.2]

$$\text{cdeg}(\mathbf{R}/(x)) = \deg(\mathcal{C}/(x\mathcal{C}) \otimes \mathbf{R}/(a)) = \deg(\mathcal{C}/(a, x)\mathcal{C}) = \deg(\mathcal{C}/(a) \otimes \mathbf{R}/(x)) \geq \text{cdeg}(\mathbf{R}).$$

\square

Question 6.12. Another question is when there exists x such that $\text{cdeg}(\mathbf{R}) = \text{cdeg}(\mathbf{R}/(x))$ in 2 cases: (i) \mathcal{C} is equimultiple and (ii) \mathcal{C} is not necessarily equimultiple. It would be a Bertini type theorem.

7. RELATIVE CANONICAL DEGREES: EXTENSIONS/VARIATIONS

It is clear that the definition of canonical degree (see Theorem 2.2) does not take into account deeper properties of \mathbf{R} . We have seen this in the case of normal rings when $\text{cdeg}(\mathbf{R}) = 0$ is all we get, no additional information about \mathbf{R} is forthcoming—except if \mathcal{C} is equimultiple. At a minimum, we would like to say that if \mathbf{R} is Cohen-Macaulay then $\text{cdeg}(\mathbf{R}) = 0$ if and only if \mathbf{R} is Gorenstein. Here are some proposals, beginning with a generalization of Proposition 2.1.

Definition 7.1. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ that admits a canonical ideal \mathcal{C} . Let $G = \text{gr}_{\mathcal{C}}(\mathbf{R}) = \bigoplus_{n \geq 0} \mathcal{C}^n / \mathcal{C}^{n+1}$ be the associated graded ring of \mathcal{C} , and $M_{\mathcal{C}} = (\mathfrak{m}, G_+)$ its maximal irrelevant ideal. The *canonical degree** of \mathbf{R} is the integer

$$\text{cdeg}_{\mathcal{C}}^*(\mathbf{R}) := e_0(M_{\mathcal{C}}, \text{gr}_{\mathcal{C}}(\mathbf{R})) - \deg(\mathbf{R}/\mathcal{C}) = \deg(\text{gr}_{\mathcal{C}}(\mathbf{R})) - \deg(\mathbf{R}/\mathcal{C}).$$

Before we discuss cases, let us recall some elementary facts about the calculation of multiplicities.

Proposition 7.2. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and let I be an ideal of positive codimension. Let Q be a reduction of I .*

- (1) *If a is a regular element, then $\deg(I/aI) = \deg(\mathbf{R}/(a))$.*
- (2) *$\deg(\text{gr}_I(\mathbf{R})) = \deg(\text{gr}_Q(\mathbf{R}))$.*
- (3) *If I is equimultiple and Q is generated by a regular sequence, then $\deg(\text{gr}_I(\mathbf{R})) = \deg(\mathbf{R}/Q)$.*

Proof. (1) Consider the two exact sequences of modules

$$\begin{aligned} 0 \rightarrow I/aI \rightarrow \mathbf{R}/aI \rightarrow \mathbf{R}/I \rightarrow 0, \\ 0 \rightarrow (a)/aI = \mathbf{R}/I \rightarrow \mathbf{R}/aI \rightarrow \mathbf{R}/(a) \rightarrow 0. \end{aligned}$$

These yield

$$\deg(\mathbf{R}/aI) = \deg(I/aI) + \deg(\mathbf{R}/I) = \deg(\mathbf{R}/I) + \deg(\mathbf{R}/(a)).$$

(2) and (3) Consider the embedding

$$0 \rightarrow \mathbf{A} = \mathbf{R}[Qt, t^{-1}] \rightarrow \mathbf{B} = \mathbf{R}[It, t^{-1}] \rightarrow L \rightarrow 0.$$

Then L is a module over \mathbf{A} of dimension $\leq d$. We consider the snake diagram defined by multiplication by t^{-1} :

$$0 \rightarrow L_1 \rightarrow \mathbf{A}/t^{-1}\mathbf{A} = \text{gr}_Q(\mathbf{R}) \rightarrow \mathbf{B}/t^{-1}\mathbf{B} = \text{gr}_I(\mathbf{R}) \rightarrow L_2 \rightarrow 0$$

where L_1 and L_2 arise from

$$0 \rightarrow L_1 \rightarrow L \xrightarrow{t^{-1}} L \rightarrow L_2 \rightarrow 0,$$

which is a nilpotent action. Note that $\dim L = \dim L_1 = \dim L_2$, and therefore $\deg(L_1) = \deg(L_2)$. The calculation of multiplicities then gives $\deg(\text{gr}_Q(\mathbf{R})) = \deg(\text{gr}_I(\mathbf{R}))$ in general, and $\deg(\text{gr}_Q(\mathbf{R})) = \deg(\mathbf{R}/Q)$ in the equimultiple case. \square

Corollary 7.3. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ that admits a canonical ideal \mathcal{C} . If \mathcal{C} is equimultiple, then $\text{cdeg}_{\mathcal{C}}^*(\mathbf{R}) = \text{cdeg}(\mathbf{R})$. In particular, $\text{cdeg}_{\mathcal{C}}^*(\mathbf{R})$ is independent of the choice of \mathcal{C} .*

Proof. Let (a) be a minimal reduction of \mathcal{C} . Then by Proposition 7.2 and Theorem 2.2, we obtain

$$\text{cdeg}_{\mathcal{C}}^*(\mathbf{R}) = \deg(\text{gr}_{\mathcal{C}}(\mathbf{R})) - \deg(\mathbf{R}/\mathcal{C}) = \deg(\mathbf{R}/(a)) - \deg(\mathbf{R}/\mathcal{C}) = \text{cdeg}(\mathbf{R}).$$

\square

Example 7.4. Define $\varphi : S = k[t_1, \dots, t_5] \rightarrow A = k[x, y]$ as $\varphi(t_i) = x^{5-i}y^{i-1}$ for each i . Let $L = \ker(\varphi)$ and $\mathbf{R} = \text{Im}(\varphi)$. Then a canonical ideal \mathcal{C} of \mathbf{R} is $\mathcal{C} = (x^3y^5, x^4y^4, x^5y^3)$ which has a minimal reduction $Q = (x^5y^3 + x^3y^5, x^4y^4)$. Using Macaulay 2, we obtain

$$\deg(\text{gr}_Q(\mathbf{R})) = 6 = \deg(\mathbf{R}/\mathcal{C})$$

Hence we have

$$\text{cdeg}_{\mathcal{C}}^*(\mathbf{R}) = \deg(\text{gr}_Q(\mathbf{R})) - \deg(\mathbf{R}/\mathcal{C}) = 0.$$

Note that \mathbf{R} is a non-Gorenstein ring of type 3.

Another family of relative cdegs arises from the following construction.

Remark 7.5. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ that admits a canonical ideal \mathcal{C} . Let $\mathbf{x} = \{x_1, \dots, x_{d-1}\} \subset \mathfrak{m} \setminus \mathfrak{m}^2$ be a minimal system of parameters for \mathbf{R}/\mathcal{C} . Consider $\text{cdeg}(\mathbf{R}/(\mathbf{x}))$. Now define

$$\text{cdeg}_{\mathcal{C}}^{\sharp}(\mathbf{R}) = \min\{\text{cdeg}(\mathbf{R}/(\mathbf{x})) \mid (\mathbf{x})\}.$$

In this case, we get $\text{cdeg}_{\mathcal{C}}^{\sharp}(\mathbf{R}) = 0$ if and only if \mathbf{R} is Gorenstein.

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